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Introduction

- In this topic, we will
  - Introduce the idea of using interpolating quadratics
  - Look at techniques to minimize error
    - Shifting to the origin
    - Using the appropriate quadratic formula
  - Discuss some of the issues in such an implementation

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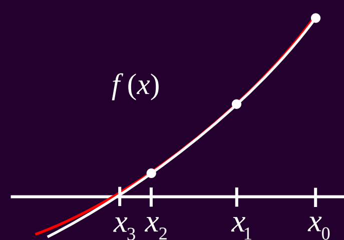
Interpolating quadratics

- The secant method interpolates two points
  - How about finding an interpolating polynomial between three points?



Interpolating quadratics

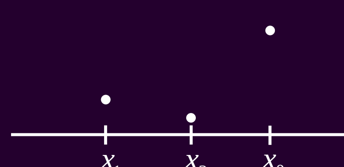
- First, rearrange the most recent three approximations so that

$$f(x_k) < f(x_{k-1}) < f(x_{k-2}) \text{ or } f(x_k) > f(x_{k-1}) > f(x_{k-2})$$

  so that $|f(x_k)| < |f(x_{k-2})|$
  - In this case, we should have

$$x_k < x_{k-1} < x_{k-2} \text{ or } x_k > x_{k-1} > x_{k-2}$$

- If this is not the case, we should revert to the secant method



Removing bracketing

- Recall there are two formulas for calculating the root of a quadratic polynomial $ax^2 + bx + c$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

  - The first is better at finding the larger root in absolute value
  - The second is better at finding the smaller root in absolute value

Removing bracketing

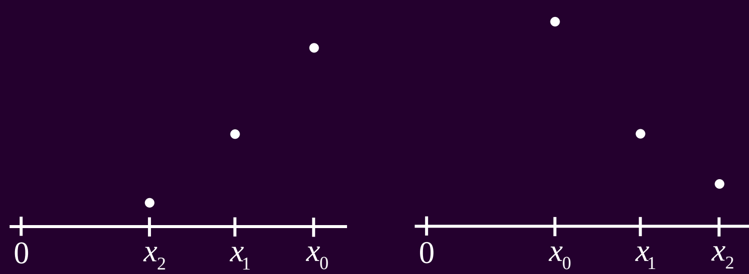
- We could find the interpolating quadratic, but this leads to an unwieldy formula:

$$\frac{\left(f(x_{k-2}) - f(x_{k-1})\right)x_k + \left(f(x_k) - f(x_{k-2})\right)x_{k-1} - \left(f(x_k) - f(x_{k-1})\right)x_{k-2}}{(x_k - x_{k-1})(x_k - x_{k-2})(x_{k-1} - x_{k-2})}x^2$$
$$+\frac{\left(f(x_{k-1}) - f(x_{k-2})\right)x_k^2 + \left(f(x_{k-2}) - f(x_k)\right)x_{k-1}^2 + x_{k-2}^2\left(f(x_k) - f(x_{k-1})\right)}{(x_k - x_{k-1})(x_k - x_{k-2})(x_{k-1} - x_{k-2})}x$$
$$+\frac{\left(f(x_{k-2})x_{k-1} - f(x_{k-1})x_{k-2}\right)x_k^2 + \left(f(x_{k-1})x_{k-2}^2 - f(x_{k-2})x_{k-1}^2\right)x_k + f(x_k)x_{k-1}x_{k-2}(x_{k-1} - x_{k-2})}{(x_k - x_{k-1})(x_k - x_{k-2})(x_{k-1} - x_{k-2})}$$

  - Also, which root do we pick: there are two

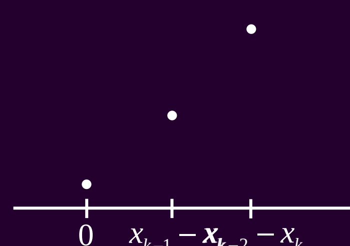
Removing bracketing

- For example,



Removing bracketing

- Strategy: Shift the x-values to the origin by x_k:



Removing bracketing

- Now we get a slightly simpler formula:

$$\frac{\left(f(x_{k-1})-f(x_{k-2})\right)x_k+\left(f(x_{k-2})-f(x_k)\right)x_{k-1}+\left(f(x_k)-f(x_{k-1})\right)x_{k-2}}{(x_{k-1}-x_{k-2})(x_{k-2}-x_k)(x_k-x_{k-1})}x^2$$

$$+\frac{\left(f(x_{k-1})-f(x_{k-2})\right)x_k^2+2\left(\left(f(x_{k-2})-f(x_k)\right)x_{k-1}+x_{k-2}\left(f(x_k)-f(x_{k-1})\right)\right)x_k-\left(f(x_{k-2})-f(x_k)\right)x_{k-1}^2-\left(f(x_k)-f(x_{k-1})\right)x_{k-2}^2}{(x_{k-1}-x_{k-2})(x_{k-2}-x_k)(x_k-x_{k-1})}x$$

$$+f(x_k)$$

  - Also, we now must find the smaller of the two roots
  - Use $\dfrac{-2c}{b \pm \sqrt{b^2-4ac}}$ and choose the sign in the denominator to make the denominator as large as possible

Rate of convergence

- If h is the error, then the rate of convergence is $O(h^{\mu})$ where μ is the real root of $x^3 = x^2 + x + 1$, so $\mu \approx 1.8393$
  - Compare this with the secant method with $\phi \approx 1.6180$

Issues

- Unlike the secant and Newton's method:
  - If the function is real-valued and the points are real, the root could be complex
  - This is useful if the polynomial has complex roots
  - This is less useful if you are trying to find a real root

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Summary

- Following this topic, you now
  - Understand Muller's method
    - It uses quadratic interpolating polynomials
  - Are aware that techniques can be used to reduce error
    - Shifting to the origin
  - Understand it may also diverge or result in complex roots, even if all values are real
  - Know that Muller's method is $O(h^{\mu})$, which converges almost as quickly as Newton's method

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References

[1] https://en.wikipedia.org/wiki/Muller%27s_method

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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